

# AVERAGES OF RATIOS OF THE RIEMANN ZETA-FUNCTION AND CORRELATIONS OF DIVISOR SUMS

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**ABSTRACT.** We establish a connection between the ratios conjecture for the Riemann zeta-function and a conjecture concerning correlations of convolutions of Möbius and divisor functions. Specifically, we prove that the ratios conjecture and an arithmetic correlations conjecture imply the same result. This provides new support for the ratios conjecture, which previously had been motivated by analogy with formulae in random matrix theory and by a heuristic recipe. Our main theorem generalises a recent calculation pertaining to the special case of two-over-two ratios.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let  $A$  and  $B$  be sets of complex numbers with real parts smaller than  $1/4$ . Let  $C$  and  $D$  be sets of complex numbers with positive real parts smaller than  $1/4$ . The purpose of this paper is to investigate the averages

$$\mathcal{R}_{A,B,C,D}(T) := \int_0^\infty \psi\left(\frac{t}{T}\right) \frac{\prod_{\alpha \in A} \zeta(s + \alpha) \prod_{\beta \in B} \zeta(1 - s + \beta)}{\prod_{\gamma \in C} \zeta(s + \gamma) \prod_{\delta \in D} \zeta(1 - s + \delta)} dt$$

where  $s = 1/2 + it$ .  $\mathcal{R}$  is the subject of the “ratios conjecture” originally formulated in [CFZ] and studied in [CS]. In these prior studies the perspective was from the point of view of analogy with Random Matrix Theory (RMT). Our new perspective is to study this quantity from an arithmetic point of view. In particular, we identify those parts of the ratios conjecture that arise from a study of the coefficient correlations

$$\sum_{n \leq X} I_{A,C}(n) I_{B,D}(n + h)$$

where  $I_{A,C}$  is defined implicitly by

$$\sum_{n=1}^{\infty} \frac{I_{A,C}(n)}{n^s} = \frac{\prod_{\alpha \in A} \zeta(s + \alpha)}{\prod_{\gamma \in C} \zeta(s + \gamma)}.$$

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In this paper we will describe this connection explicitly.

Not surprisingly,  $\mathcal{R}$  is related to averages of the (analytic continuation of the) Rankin-Selberg convolution

$$\mathcal{B}_{A,B,C,D}(s) := \sum_{n=1}^{\infty} \frac{I_{A,C}(n)I_{B,D}(n)}{n^s}.$$

In fact, we can state the ratios conjecture in a relatively simple way in terms of  $\mathcal{B}$ .

**Conjecture 1.** ([CFZ] and [CS]) *Suppose that the sets  $A, B, C$  and  $D$  are as in the introduction and that the imaginary parts of all of the parameters in this set are  $O(T^{1-\xi})$  for some  $\xi > 0$ . Then*

$$\mathcal{R}_{A,B,C,D}(T) = \int_0^\infty \psi\left(\frac{t}{T}\right) \sum_{\substack{U \subset A, V \subset B \\ |U|=|V|}} \left(\frac{t}{2\pi}\right)^{-\sum_{\hat{a} \in U} \hat{a} - \sum_{\hat{b} \in V} \hat{b}} \mathcal{B}_{A-U+V^-, B-V+U^-, C, D}(1) dt + O(T^{1-\eta})$$

for some  $\eta > 0$ .

Here  $V^-$  denotes the set obtained from  $V$  by replacing every element by its negative. So, the set  $A - U + V^-$  may be obtained from the set  $A$  by deleting the elements of  $U$  and then inserting the negatives of the elements of  $V$ . (We assume that the elements of all of these sets are distinct. The situation with repeating elements may be deduced from this case by a limiting argument.)

It is also not surprising that  $\mathcal{R}$  is connected to weighted averages over  $n$  and  $h$  of

$$I_{A,C}(n)I_{B,D}(n+h).$$

It is this connection that we are elucidating.

Using the  $\delta$ -method [DFI] it transpires that the weighted averages relevant to the consideration of  $\mathcal{R}$  may be expressed in terms of

$$\begin{aligned} \mathcal{C}_{A,B,C,D}(s) &:= \frac{1}{(2\pi i)^2} \int_{|w-1|=\epsilon} \int_{|z-1|=\epsilon} \chi(w+z-s-1) \sum_{q=1}^{\infty} \sum_{h=1}^{\infty} \frac{r_q(h)}{h^{s+2-w-z}} \\ &\quad \times \sum_{m=1}^{\infty} \frac{I_{A,C}(m)e(m/q)}{m^w} \sum_{n=1}^{\infty} \frac{I_{B,D}(n)e(n/q)}{n^z} dw dz \end{aligned}$$

where  $r_q(h)$  denotes Ramanujan's sum and where  $\chi(s)$  is the factor from the functional equation  $\zeta(s) = \chi(s)\zeta(1-s)$ ; also here and elsewhere  $\epsilon$  is chosen to be larger than the absolute values of the shift parameters  $\alpha, \beta, \gamma, \delta$  but smaller than  $1/2$ .

The main conclusion of this paper is that the arithmetic contributions arising from the averages of  $I_{A,C}(n)I_{B,D}(n+h)$  coincide exactly with the terms from the ratios conjecture with  $|U| = |V| = 1$ , i.e. what are referred to elsewhere as the “one-swap” terms.

The result that explicates this is encapsulated in the following identity.

**Theorem 1.** *Assuming the Generalized Riemann Hypothesis*

$$\mathcal{C}_{A,B,C,D}(s) = \sum_{\substack{U \subset A, V \subset B \\ |U|=|V|=1}} \mathcal{B}_{A-U+V^-, B-V+U^-, C, D}(s+1).$$

It turns out to be convenient to study an average of the ratios conjecture. To this end let

$$\mathcal{I}_{A,C}(s; X) = \sum_{n \leq X} I_{A,C}(n) n^{-s}.$$

We are interested in the average over  $t$  of  $\mathcal{I}_{A,C}(s, X) \overline{\mathcal{I}_{B,D}(1-s, X)}$  (N.B.  $s = 1/2 + it$ ) in the case that  $X = T^\lambda$  for some  $\lambda > 1$ . (When  $\lambda < 1$  this average is dominated by diagonal terms.) We give two different treatments of the average of “truncated” ratios:

$$\mathcal{M}_{A,B,C,D}(T; X) := \int_0^\infty \psi\left(\frac{t}{T}\right) \mathcal{I}_{A,C}(s, X) \overline{\mathcal{I}_{B,D}(1-s, X)} dt$$

(where again  $s = 1/2 + it$ ) which lead to the same answer. The first is by the ratios conjecture and the second is by consideration of the correlations of the coefficients.

In each case we prove

**Theorem 2.** *Let  $A, B, C, D$  be as above. Then, assuming either a uniform version of the ratios conjecture or a uniform version of the conjectural formula for correlations of values of  $I_{\alpha, \gamma}(n)$ , we have for some  $\eta > 0$  and some  $\lambda > 1$ ,*

$$\begin{aligned} \mathcal{M}_{A,B,C,D}(T; X) = & \int_0^\infty \psi\left(\frac{t}{T}\right) \frac{1}{2\pi i} \int_{\Re s=2} \sum_{\substack{U \subset A, V \subset B \\ |U|=|V| \leq 1}} \left(\frac{t}{2\pi}\right)^{-|U|s - \sum_{\hat{\alpha} \in U} \hat{\alpha} - \sum_{\hat{\beta} \in V} \hat{\beta}} \mathcal{B}_{A-U+V^-, B-V+U^-, C, D}(s+1) \frac{X^s}{s} ds dt \\ & + O(T^{1-\eta}). \end{aligned}$$

This shows that the ratios conjecture follows not only from the ‘recipe’ of [CFZ], but also relates to correlations of values of  $I_{A,C}(n)$ .

An earlier paper [CK5] had this calculation but in the special case that all of the sets  $A, B, C, D$  are singletons. That paper has additional background information and motivation, in particular relating to connections with correlations between the zeros (c.f. [BK1, BK2]) and with the moments (c.f. [CK1, CK2, CK3, CK4]) of the zeta function. The first part of the present calculation follows closely that in [CK5].

## 2. APPROACH VIA THE RATIOS CONJECTURE

We have

$$\mathcal{I}_{A,C}(s, X) = \frac{1}{2\pi i} \int_{(2)} \mathcal{I}_{A,C}(s+w) \frac{X^w}{w} dw;$$

there is a similar expression for  $\mathcal{I}_{B,D}(s, X)$ . Inserting these expressions and rearranging the integrations we have

$$\mathcal{M}_{A,B,C,D}(T; X) = \frac{1}{(2\pi i)^2} \int_{\Re w=2} \int_{\Re z=2} \frac{X^{w+z}}{wz} \mathcal{R}_{A_w, B_z, C_w, D_z}(T) dw dz.$$

We note that Conjecture 1 implies that  $\mathcal{R}_{A_w, B_z, C_w, D_z}$  is, to leading order as  $T \rightarrow \infty$ , a function of  $z + w$ . We therefore make the change of variable  $s = z + w$ ; now the integration in the  $s$  variable is on the vertical line  $\Re s = 4$ . We retain  $z$  as our other variable and integrate over it. This turns out to be the integral

$$\frac{1}{2\pi i} \int_{\Re z=2} \frac{dz}{z(s-z)} = \frac{1}{s}$$

as is seen by moving the path of integration to the left to  $\Re z = -\infty$ . Thus we have that  $\mathcal{M}_{A,B,C,D}(T; X)$  is given to leading order by

$$\frac{1}{2\pi i} \int_{\Re s=4} \frac{X^s}{s} \mathcal{R}_{A_s, B, C_s, D}(T) ds.$$

Moving the path of integration to  $\Re s = \epsilon$ , avoiding any poles, inserting Conjecture 1, and noting that

$$\mathcal{B}_{A_s, B, C_s, D}(1) = \mathcal{B}_{A, B, C, D}(s+1),$$

we have that the uniform ratios conjecture implies the conclusion of Theorem 2.

### 3. APPROACH VIA COEFFICIENT CORRELATIONS

We follow the approach developed by Goldston and Gonek [GG] in their work on mean-values of long Dirichlet polynomials.

Expanding the sums and integrating term-by-term, we have

$$\mathcal{M}_{\alpha, \beta, \gamma, \delta}(T; X) = T \sum_{m, n \leq X} \frac{I_{A,C}(m) I_{B,D}(n)}{\sqrt{mn}} \hat{\psi} \left( \frac{T}{2\pi} \log \frac{m}{n} \right).$$

**3.1. Diagonal.** The diagonal term is

$$T \hat{\psi}(0) \sum_{m \leq X} \frac{I_{A,C}(m) I_{B,D}(m)}{m}.$$

By Perron's formula this sum is

$$\frac{1}{2\pi i} \int_{(2)} \mathcal{B}_{A,B,C,D}(s+1) \frac{X^s}{s} ds.$$

**3.2. Off-diagonal.** For the off-diagonal terms we need to analyze

$$2T \sum_{T \leq m \leq X} \sum_{1 \leq h \leq \frac{X}{T}} \frac{I_{A,C}(m) I_{B,D}(m+h)}{m} \hat{\psi} \left( \frac{Th}{2\pi m} \right).$$

We replace the arithmetic terms by their average and express this as

$$2T \int_T^X \sum_{1 \leq h \leq \frac{X}{T}} \frac{\langle I_{A,C}(m) I_{B,D}(m+h) \rangle_{m \sim u}}{u} \hat{\psi} \left( \frac{Th}{2\pi u} \right) du.$$

We now compute the average heuristically via the delta-method [DFI]:

$$\langle I_{A,C}(m) I_{B,D}(m+h) \rangle_{m \sim u} \sim \sum_{q=1}^{\infty} r_q(h) \langle I_{A,C}(m) e(m/q) \rangle_{m \sim u} \langle I_{B,D}(m) e(m/q) \rangle_{m \sim u}$$

where  $r_q(h)$  is the Ramanujan sum, a formula for which is  $r_q(h) = \sum_{\substack{d|h \\ d|q}} d \mu(\frac{q}{d})$ . This may be formalized as a precise conjecture exactly as in Section 5 of [CK5]. It is this conjectural formula that we refer to in Theorem 2. Now

$$\langle I_{A,C}(m) e(m/q) \rangle_{m \sim u} = \frac{1}{2\pi i} \int_{|w-1|=\epsilon} \sum_{m=1}^{\infty} I_{A,C}(m) e(m/q) m^{-w} u^{w-1} dw.$$

The off-diagonal contribution is thus

$$\begin{aligned} & 2T \sum_{1 \leq h \leq \frac{X}{T}} \int_T^X \frac{1}{(2\pi i)^2} \iint_{\substack{|w-1|=\epsilon \\ |z-1|=\epsilon}} \sum_{q=1}^{\infty} r_q(h) \hat{\psi} \left( \frac{Th}{2\pi u} \right) u^{w+z-2} \\ & \times \sum_{m_1=1}^{\infty} \frac{I_{A,C}(m_1) e(m_1/q)}{m_1^w} \sum_{m_2=1}^{\infty} \frac{I_{B,D}(m_2) e(m_2/q)}{m_2^z} dw dz \frac{du}{u}. \end{aligned}$$

We next make the change of variables  $v = \frac{Th}{2\pi u}$ . The inequality  $u \leq X$  then implies that  $\frac{Th}{2\pi v} \leq X$  or  $h \leq \frac{2\pi v X}{T}$ . The above can be re-expressed as

$$\begin{aligned} & 2T \int_0^{\infty} \sum_{1 \leq h \leq \frac{2\pi v X}{T}} \frac{1}{(2\pi i)^2} \iint_{\substack{|w-1|=\epsilon \\ |z-1|=\epsilon}} \sum_{q=1}^{\infty} r_q(h) \hat{\psi}(v) \left( \frac{Th}{2\pi v} \right)^{w+z-2} \\ & \times \sum_{m_1=1}^{\infty} \frac{I_{A,C}(m_1) e(m_1/q)}{m_1^w} \sum_{m_2=1}^{\infty} \frac{I_{B,D}(m_2) e(m_2/q)}{m_2^z} dw dz \frac{dv}{v}. \end{aligned}$$

Using Perron's formula to express the sum over  $h$  gives

$$2T \int_0^\infty \frac{1}{(2\pi i)^3} \int_{\Re s=2} \iint_{\substack{|w-1|=\epsilon \\ |z-1|=\epsilon}} \sum_{q=1}^\infty \sum_{h=1}^\infty \frac{r_q(h)}{h^s} \hat{\psi}(v) \left( \frac{Th}{2\pi v} \right)^{w+z-2} \left( \frac{2\pi v X}{T} \right)^s \\ \times \sum_{m_1=1}^\infty \frac{I_{A,C}(m_1)e(m_1/q)}{m_1^w} \sum_{m_2=1}^\infty \frac{I_{B,D}(m_2)e(m_2/q)}{m_2^z} \frac{ds}{s} dw dz \frac{dv}{v}.$$

Now

$$2 \int_0^\infty \hat{\psi}(v) v^A \frac{dv}{v} = \chi(1-A) \int_0^\infty \psi(t) t^{-A} dt.$$

Incorporating this formula gives

$$T \int_0^\infty \psi(t) \frac{1}{(2\pi i)^3} \int_{\Re s=2} \iint_{\substack{|w-1|=\epsilon \\ |z-1|=\epsilon}} \sum_{q=1}^\infty \sum_{h=1}^\infty \frac{r_q(h)}{h^{s+2-w-z}} \left( \frac{Tt}{2\pi} \right)^{w+z-2} \left( \frac{2\pi X}{tT} \right)^s \chi(w+z-s-1) \\ \times \sum_{m_1=1}^\infty \frac{I_{A,C}(m_1)e(m_1/q)}{m_1^w} \sum_{m_2=1}^\infty \frac{I_{B,D}(m_2)e(m_2/q)}{m_2^z} \frac{ds}{s} dw dz dt.$$

By Theorem 1, this is

$$\sum_{\substack{\hat{\alpha} \in A \\ \hat{\beta} \in B}} \int_0^\infty \psi \left( \frac{t}{T} \right) \frac{1}{2\pi i} \int_{\Re s=2} \left( \frac{t}{2\pi} \right)^{-\hat{\alpha}-\hat{\beta}-s} \mathcal{B}_{A' \cup \{-\hat{\beta}\}, B' \cup \{-\hat{\alpha}\}, C, D}(s+1) \frac{X^s}{s} ds dt$$

where  $A' = A - \{\hat{\alpha}\}$  and  $B' = B - \{\hat{\beta}\}$ . Adding the diagonal and off-diagonal terms, we thus obtain that the conjecture for the correlations of values of  $I_{A,C}(n)$  also implies the conclusion of Theorem 2.

#### 4. PROOF OF THEOREM 1

First of all, we have

$$\sum_{h=1}^\infty \frac{r_q(h)}{h^A} = \sum_{h=1}^\infty \frac{\sum_{g|h} \frac{g}{h} \mu\left(\frac{q}{g}\right)}{h^A} = \sum_{g|q} g^{1-A} \mu\left(\frac{q}{g}\right) \zeta(A) = q^{1-A} \Phi(1-A, q) \zeta(A)$$

where

$$\Phi(x, q) = \prod_{p|q} \left( 1 - \frac{1}{p^x} \right).$$

Using this and the functional equation for  $\zeta$ , we have to evaluate

$$\begin{aligned} & \frac{1}{(2\pi i)^2} \iint_{\substack{|w-1|=\epsilon \\ |z-1|=\epsilon}} \sum_{q=1}^{\infty} q^{w+z-s-1} \Phi(w+z-s-1, q) \\ & \times \zeta(w+z-s-1) \sum_{m_1=1}^{\infty} \frac{I_{A,C}(m_1)e(m_1/q)}{m_1^w} \sum_{m_2=1}^{\infty} \frac{I_{B,D}(m_2)e(m_2/q)}{m_2^z} dw dz. \end{aligned}$$

We identify the polar structure of the Dirichlet series here by passing to characters via the formula

$$e\left(\frac{m}{q}\right) = \sum_{\substack{d|m \\ d|q}} \frac{1}{\phi\left(\frac{q}{d}\right)} \sum_{\chi \bmod \frac{q}{d}} \tau(\bar{\chi}) \chi\left(\frac{m}{d}\right).$$

Assuming GRH, the only poles near  $w = 1$  arise from the principal characters  $\chi_{\frac{q}{d}}^{(0)}$ . Using

$$\tau(\chi_{\frac{q}{d}}^{(0)}) = \mu\left(\frac{q}{d}\right)$$

we have that the poles of  $\sum_{m=1}^{\infty} I_{A,C}(m)e(m/q)m^{-w}$  are the same as the poles of

$$\begin{aligned} & \sum_{d|q} \frac{\mu\left(\frac{q}{d}\right)}{\phi\left(\frac{q}{d}\right)} \sum_{m=1}^{\infty} I_{A,C}(md) \chi_{\frac{q}{d}}^{(0)}(m) m^{-w} d^{-w} \\ & = q^{-w} \sum_{d|q} \frac{\mu(d)}{\phi(d)} d^w \sum_{m=1}^{\infty} \frac{I_{A,C}\left(\frac{mq}{d}\right) \chi_d^{(0)}(m)}{m^w} \end{aligned}$$

and the principal parts are the same. We now replace  $\chi_d^{(0)}(m)$  by  $\sum_{\substack{e|d \\ e|m}} \mu(e)$ . This leads to

$$q^{-w} \sum_{d|q} \frac{\mu(d) d^w}{\phi(d)} \sum_{e|d} \mu(e) e^{-w} \sum_{m=1}^{\infty} \frac{I_{A,C}\left(\frac{meq}{d}\right)}{m^w}.$$

Now we need the polar structure of

$$\sum_{m=1}^{\infty} I_{A,C}(mr) m^{-w}$$

for  $r = qe/d$ .

Since  $I_{A,C}(n)$  is a multiplicative function of  $n$ ,  $I_{A,C}(nr)/I_{A,C}(r)$  is also a multiplicative function of  $n$ . The generating function may therefore be expressed as an Euler product:

$$\sum_{n=1}^{\infty} \frac{I_{A,C}(nr)/I_{A,C}(r)}{n^w} = \sum_{n=1}^{\infty} \frac{I_{A,C}(n)}{n^w} \prod_{p|r} \frac{\sum_{j=0}^{\infty} \frac{I_{A,C}(p^{j+\lambda_r(p)})/I_{A,C}(p^{\lambda_r(p)})}{p^{jw}}}{\sum_{j=0}^{\infty} \frac{I_{A,C}(p^j)}{p^{jw}}}$$

This gives

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{I_{A,C}(nr)}{n^w} &= \frac{\prod_{\alpha \in A} \zeta(w + \alpha)}{\prod_{\gamma \in C} \zeta(w + \gamma)} \prod_{p|r} \frac{\sum_{j=0}^{\infty} \frac{I_{A,C}(p^{j+\lambda_r(p)})}{p^{jw}}}{\sum_{j=0}^{\infty} \frac{I_{A,C}(p^j)}{p^{jw}}} \\ &= \frac{\prod_{\alpha \in A} \zeta(w + \alpha)}{\prod_{\gamma \in C} \zeta(w + \gamma)} E_{A,C}(w, r), \end{aligned}$$

say. In particular, the poles are at  $w = 1 - \alpha$  for  $\alpha \in A$ . Thus, the integral over  $w$  and  $z$  is

$$\begin{aligned} &\sum_{\substack{\hat{\alpha} \in A \\ \hat{\beta} \in B}} \sum_{q=1}^{\infty} q^{1-\hat{\alpha}-\hat{\beta}-s} \Phi(1 - \hat{\alpha} - \hat{\beta} - s, q) \zeta(1 - \hat{\alpha} - \hat{\beta} - s) \\ &\quad \times q^{-1+\hat{\alpha}} \sum_{d_1|q} \frac{\mu(d_1) d_1^{1-\hat{\alpha}}}{\phi(d_1)} \sum_{e_1|d_1} \mu(e_1) e_1^{-1+\hat{\alpha}} q^{-1+\hat{\beta}} \sum_{d_2|q} \frac{\mu(d_2) d_2^{1-\hat{\beta}}}{\phi(d_2)} \sum_{e_2|d_2} \mu(e_2) e_2^{-1+\hat{\beta}} \\ &\quad \times \frac{\prod_{\alpha \in A'} \zeta(1 - \hat{\alpha} + \alpha) \prod_{\beta \in B'} \zeta(1 - \hat{\beta} + \beta)}{\prod_{\gamma \in C} \zeta(1 - \hat{\alpha} + \gamma) \prod_{\delta \in D} \zeta(1 - \hat{\beta} + \delta)} E_{A,C}(1 - \hat{\alpha}, \frac{qe_1}{d_1}) E_{B,D}(1 - \hat{\beta}, \frac{qe_2}{d_2}). \end{aligned}$$

So we have to identify the Dirichlet series

$$\begin{aligned} &\sum_{q=1}^{\infty} q^{-1-s} \Phi(1 - \hat{\alpha} - \hat{\beta} - s, q) \mathcal{E}_{A,C}(1 - \hat{\alpha}, q) \mathcal{E}_{B,D}(1 - \hat{\beta}, q) \\ &= \sum_{r=1}^{\infty} \frac{\mu(r)}{r^{2-\hat{\alpha}-\hat{\beta}}} \sum_{q=1}^{\infty} \frac{\mathcal{E}_{A,C}(1 - \hat{\alpha}, qr) \mathcal{E}_{B,D}(1 - \hat{\beta}, qr)}{q^{1+s}} \end{aligned}$$

where we have made use of  $\Phi(\xi, q) = \sum_{r|q} \mu(r) r^{-\xi}$ , and where

$$\mathcal{E}_{A,C}(1 - \hat{\alpha}, q) = \sum_{d|q} \frac{\mu(d) d^{1-\hat{\alpha}}}{\phi(d)} \sum_{e|d} \mu(e) e^{-1+\hat{\alpha}} E_{A,C}(1 - \hat{\alpha}, \frac{qe}{d}).$$

This itself may be expressed as an Euler product. So, let's assume  $q = p^J$  with  $J \geq 1$  and identify

$$\begin{aligned} &\sum_{d|q} \frac{\mu(d) d^{1-\hat{\alpha}}}{\phi(d)} \sum_{e|d} \mu(e) e^{-1+\hat{\alpha}} E_{A,C}(1 - \hat{\alpha}, \frac{qe}{d}) \\ &= E_{A,C}(1 - \hat{\alpha}, p^J) - \frac{p^{1-\hat{\alpha}}}{p-1} E_{A,C}(1 - \hat{\alpha}, p^{J-1}) + \frac{1}{p-1} E_{A,C}(1 - \hat{\alpha}, p^J) \\ &= \frac{p}{p-1} E_{A,C}(1 - \hat{\alpha}, p^J) - \frac{p^{1-\hat{\alpha}}}{p-1} E_{A,C}(1 - \hat{\alpha}, p^{J-1}). \end{aligned}$$

Now we note the identity

$$I_{A,C}(p^J) = I_{A',C}(p^J) + p^{-\alpha} I_{A,C}(p^{J-1})$$



where  $A = A' \cup \{\alpha\}$ . Thus

$$\sum_{j=0}^{\infty} \frac{I_{A,C}(p^{j+J})}{p^{jw}} - p^{-\alpha} \sum_{j=0}^{\infty} \frac{I_{A,C}(p^{j+J-1})}{p^{jw}} = \sum_{j=0}^{\infty} \frac{I_{A',C}(p^{j+J})}{p^{jw}}.$$

Thus,

$$\begin{aligned} \mathcal{E}_{A,C}(1 - \hat{\alpha}, p^J) &= \frac{p}{p-1} \frac{\sum_{j=0}^{\infty} \frac{I_{A',C}(p^{j+J})}{p^{j(1-\hat{\alpha})}}}{\sum_{j=0}^{\infty} \frac{I_{A,C}(p^j)}{p^{j(1-\hat{\alpha})}}} \\ &= \frac{p}{p-1} \frac{\prod_{\alpha \in A} (1 - \frac{1}{p^{1-\hat{\alpha}+\alpha}})}{\prod_{\gamma \in C} (1 - \frac{1}{p^{1-\hat{\alpha}+\gamma}})} \sum_{j=0}^{\infty} \frac{I_{A',C}(p^{j+J})}{p^{j(1-\hat{\alpha})}} \\ &= \frac{\prod_{\alpha \in A'} (1 - \frac{1}{p^{1-\hat{\alpha}+\alpha}})}{\prod_{\gamma \in C} (1 - \frac{1}{p^{1-\hat{\alpha}+\gamma}})} \sum_{j=0}^{\infty} \frac{I_{A',C}(p^{j+J})}{p^{j(1-\hat{\alpha})}}. \end{aligned}$$

Now, all that is left to do is to prove that

$$\sum_{\ell=0}^{\infty} \frac{\mu(p^\ell)}{p^{\ell(2-\hat{\alpha}-\hat{\beta})}} \sum_{J=0}^{\infty} \frac{1}{p^J} \sum_{j=0}^{\infty} \frac{I_{A',C}(p^{j+\ell+J})}{p^{j(1-\hat{\alpha})}} \sum_{k=0}^{\infty} \frac{I_{B',D}(p^{k+\ell+J})}{p^{k(1-\hat{\beta})}} = \left(1 - \frac{1}{p^{1-\hat{\alpha}-\hat{\beta}}}\right) \sum_{\ell=0}^{\infty} \frac{I_{A' \cup \{-\hat{\beta}\},C}(p^\ell) I_{B' \cup \{-\alpha\},D}(p^\ell)}{p^\ell}$$

Temporarily let  $X = 1/p$ ,  $Y = p^{-\hat{\alpha}}$ ,  $Z = p^{-\hat{\beta}}$ ,  $a_j = I_{A,C}(p^j)$ ,  $a'_j = I_{A',C}(p^j)$ ,  $\tilde{a}_j = I_{A' \cup \{-\hat{\beta}\},C}(p^j)$ ; and  $b_k = I_{B,D}(p^k)$ ,  $b'_k = I_{B',D}(p^k)$ ,  $\tilde{b}_k = I_{B' \cup \{-\hat{\alpha}\},D}(p^k)$ . Then the desired identity follows from the theorem of the next section.

## 5. THE IDENTITY

**Theorem 3.** *Suppose that  $a', b', \tilde{a}, \tilde{b}$  are sequences such that*

$$\sum_{J=0}^{\ell} Z^{J-\ell} a'_J = \tilde{a}_\ell \quad \sum_{K=0}^{\ell} Y^{K-\ell} b'_K = \tilde{b}_\ell.$$

Then

$$\sum_{J=0}^{\infty} \sum_{\ell=0}^{\min(1,J)} (-1)^\ell \frac{X^{2\ell+J}}{Y^\ell Z^\ell} \sum_{j=0}^{\infty} a'_{j+\ell+J} \left(\frac{X}{Y}\right)^j \sum_{k=0}^{\infty} b'_{k+\ell+J} \left(\frac{X}{Z}\right)^k = \left(1 - \frac{X}{YZ}\right) \sum_{\ell=0}^{\infty} \tilde{a}_\ell \tilde{b}_\ell X^\ell.$$

*Proof.* The left side may be written as

$$\begin{aligned} &\sum_{J=0}^{\infty} X^J \sum_{j=0}^{\infty} a'_{j+J} \left(\frac{X}{Y}\right)^j \sum_{k=0}^{\infty} b'_{k+J} \left(\frac{X}{Z}\right)^k \\ &\quad - \sum_{J=0}^{\infty} X^J \sum_{j=0}^{\infty} a'_{j+1+J} \left(\frac{X}{Y}\right)^{j+1} \sum_{k=0}^{\infty} b'_{k+1+J} \left(\frac{X}{Z}\right)^{k+1} \end{aligned}$$

This may be rewritten as

$$\begin{aligned} \sum_{J=0}^{\infty} X^J & \left[ \left( \sum_{j=0}^{\infty} a'_{j+J} \left( \frac{X}{Y} \right)^j - \sum_{j=0}^{\infty} a'_{j+1+J} \left( \frac{X}{Y} \right)^{j+1} \right) \sum_{k=0}^{\infty} b'_{k+J} \left( \frac{X}{Z} \right)^k \right. \\ & \left. + \sum_{j=0}^{\infty} a'_{j+1+J} \left( \frac{X}{Y} \right)^{j+1} \left( \sum_{k=0}^{\infty} b'_{k+J} \left( \frac{X}{Z} \right)^k - \sum_{k=0}^{\infty} b'_{k+1+J} \left( \frac{X}{Z} \right)^{k+1} \right) \right] \end{aligned}$$

which simplifies to

$$\sum_{J=0}^{\infty} X^J \left[ a'_J \sum_{k=0}^{\infty} b'_{k+J} \left( \frac{X}{Z} \right)^k + b'_J \sum_{j=0}^{\infty} a'_{j+1+J} \left( \frac{X}{Y} \right)^{j+1} \right]$$

Now

$$\sum_{J=0}^{\infty} X^J a'_J \sum_{k=0}^{\infty} b'_{k+J} \left( \frac{X}{Z} \right)^k = \sum_{\ell=0}^{\infty} X^{\ell} b'_{\ell} \sum_{J=0}^{\ell} a'_J Z^{J-\ell} = \sum_{\ell=0}^{\infty} \tilde{a}_{\ell} b'_{\ell} X^{\ell}.$$

And

$$\begin{aligned} & \sum_{J=0}^{\infty} X^J b'_J \sum_{j=0}^{\infty} a'_{j+1+J} \left( \frac{X}{Y} \right)^{j+1} \\ &= \sum_{J=0}^{\infty} X^J b'_J \sum_{j=0}^{\infty} a'_{j+J} \left( \frac{X}{Y} \right)^j - \sum_{J=0}^{\infty} a'_J b'_J X^J \\ &= \sum_{\ell=0}^{\infty} a'_{\ell} \tilde{b}_{\ell} X^{\ell} - \sum_{\ell=0}^{\infty} a'_{\ell} b'_{\ell} X^{\ell}. \end{aligned}$$

Thus, the left side of the identity is

$$\sum_{\ell=0}^{\infty} \tilde{a}_{\ell} b'_{\ell} X^{\ell} + \sum_{\ell=0}^{\infty} a'_{\ell} \tilde{b}_{\ell} X^{\ell} - \sum_{\ell=0}^{\infty} a'_{\ell} b'_{\ell} X^{\ell}.$$

But  $a'_{\ell} = \tilde{a}_{\ell} - \frac{\tilde{a}_{\ell-1}}{Z}$  and  $b'_{\ell} = \tilde{b}_{\ell} - \frac{\tilde{b}_{\ell-1}}{Y}$  so that

$$\tilde{a}_{\ell} b'_{\ell} + a'_{\ell} \tilde{b}_{\ell} - a'_{\ell} b'_{\ell} = \tilde{a}_{\ell} \tilde{b}_{\ell} - \frac{\tilde{a}_{\ell-1} \tilde{b}_{\ell-1}}{YZ}.$$

The sum over  $\ell$  of this expression times  $X^{\ell}$  gives the right side of the identity.  $\square$

The reader may have noticed the similarity between this identity and the corresponding identity that formed the crux of [CK3].

# REFERENCES

- [BK1] E. B. Bogomolny and J. P. Keating. Random matrix theory and the Riemann zeros I: three- and four-point correlations. *Nonlinearity* 8 (1995), 1115–1131.
- [BK2] E. B. Bogomolny and J. P. Keating. Random matrix theory and the Riemann zeros II:  $n$ -point correlations. *Nonlinearity* 9 (1996), 911–935.
- [CFZ] J. B. Conrey, D. W. Farmer and M. R. Zirnbauer. Autocorrelation of ratios of  $L$ -functions. *Commun. Number Theory Phys.* 2 (2008), 593–636.
- [CK1] J.B. Conrey and J.P. Keating. Moments of zeta and correlations of divisor-sums: I. *Phil. Trans. R. Soc. A* 373 (2015), 20140313; arXiv:1506.06842
- [CK2] J.B. Conrey and J.P. Keating. Moments of zeta and correlations of divisor-sums: II. In *Advances in the Theory of Numbers – Proceedings of the Thirteenth Conference of the Canadian Number Theory Association*, Fields Institute Communications (Editors: A. Alaca, S. Alaca & K.S. Williams), 75–85 (2015, Springer); arXiv:1506.06843
- [CK3] J.B. Conrey and J.P. Keating. Moments of zeta and correlations of divisor-sums: III. *Indagationes Mathematicae* 26 (2015), no. 5, 736–747; arXiv:1506.06844
- [CK4] J.B. Conrey and J.P. Keating. Moments of zeta and correlations of divisor-sums: IV. *Res. Number Theory* 2 (2016), 2:24; arXiv:1506.06844
- [CK5] J. B. Conrey and J. P. Keating. Pair correlation and twin primes revisited. *Proc. R. Soc. A* 472 (2016), 20160548; arXiv:1604.06124
- [CSn] J. B. Conrey and N. C. Snaith. Applications of the  $L$ -functions ratios conjectures. *Proc. Lond. Math. Soc.* (3) 94 (2007), no. 3, 594–646.
- [DFI] W. Duke, J. B. Friedlander, and H. Iwaniec. A quadratic divisor problem. *Invent. Math.* 115 (1994), no. 2, 209–217.
- [GG] D. A. Goldston and S. M. Gonek. Mean value theorems for long Dirichlet polynomials and tails of Dirichlet series. *Acta Arith.* 84 (1998), no. 2, 155–192.

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